

# LIE TRIPLE SYSTEM CENTRAL EXTENSIONS OF LIE ALGEBRAS

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## INTRODUCTION.

The present paper deals with a certain interrelationship between Lie algebras, Lie triple systems and Leibniz algebras which shows up through the notion of universal central extension.

Lie algebras are classical algebraic structures very widely used in mathematics and physics. Another, less widely used but equally classical structures are Lie triple systems (see [1, 2, 3, 4]). They are also used in different areas of mathematics. Leibniz algebra is yet another algebraic structure gaining increasing importance. It is a sort of “noncommutative” generalization of the Lie algebra structure recently introduced in [5]. Original reason for introduction of Leibniz algebras was a new cohomology theory (see [6, 7]). Later, the notion of Leibniz algebra found other applications in different areas of mathematics.

It is well known that on each Lie algebra one may define canonically the structure of a Lie triple system. We use Leibniz algebras to study Lie triple systems arising in this way, in a similar spirit to [8] where Leibniz algebras were applied to the study of Lie algebras.

Namely, it is known that each Lie algebra is also a particular case of a Leibniz algebra. Thus speaking of universal central extensions, each perfect Lie algebra  $\mathfrak{g}$ , along with its universal central extension as a Lie algebra  $U_{Lie}(\mathfrak{g})$ , also admits a *Lie triple system universal central extension*  $U_{LTS}(\mathfrak{g})$  (if we consider  $\mathfrak{g}$  as a Lie triple system) as well as the *Leibniz universal central extension*  $U_{Leib}(\mathfrak{g})$  (if we consider  $\mathfrak{g}$  as a Leibniz algebra).

Now a nice unexpected new fact is that the Lie triple system  $U_{LTS}(\mathfrak{g})$  turns out to carry a natural Leibniz algebra structure (see Lemma 1). This fact allows us to use properties of Leibniz and Lie algebras for Lie triple systems. Namely, the main purpose of the present article is to prove the following theorem:

**Theorem 1.** *For a perfect Lie algebra  $\mathfrak{g}$  over a field  $F$ , we have the isomorphisms*

$$U_{LTS}(\mathfrak{g}) \cong U_{Leib}(\mathfrak{g}), \quad \text{if } \text{char } F = 2,$$

$$U_{LTS}(\mathfrak{g}) \cong U_{Lie}(\mathfrak{g}), \quad \text{if } \text{char } F \neq 2.$$

*In other words, the universal central extension of  $\mathfrak{g}$  in the category of Lie triple systems is isomorphic either to the universal central extension of  $\mathfrak{g}$  as a Lie algebra (if  $\text{char } F \neq 2$ ) or to the universal central extension of  $\mathfrak{g}$  as a Leibniz algebra (if  $\text{char } F = 2$ ).*

In the section 1 we recall some well known facts about Lie algebras, Leibniz algebras and Lie triple systems. In particular, in 1.2 we introduce non-abelian tensor product of Lie triple systems and in 1.3 we give general definitions and standard facts about universal central extensions of Lie triple systems. The key lemma about natural Leibniz algebra structures on Lie triple system central extensions of perfect Lie algebras is proved in the section 2. Finally, using this lemma, we obtain the proof of the main theorem (Theorem 1) in section 3.

Throughout this paper  $F$  is a field. All vector spaces and homomorphisms are considered over  $F$  if not specified otherwise.

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## 1. DEFINITIONS AND CONSTRUCTIONS.

**1.1. Recollections.** Let us begin by recalling some well known definitions and facts. Namely, we discuss two generalizations of the notion of a Lie algebra — the notions of a Lie triple system and a Leibniz algebra. We also study a relationship between these two generalizations.

The notion of a Lie triple system is a classical one. The definition of Lie triple systems can be found in many articles, but for the convenience of the reader we recall it here.

**Definition 1.** A *Lie triple system* is a vector space  $L$  equipped with a ternary bracket

$$\{-, -, -\} : L \otimes L \otimes L \rightarrow L,$$

satisfying the following

$$\begin{aligned} \{x, y, y\} &= 0, \\ \{x, y, z\} + \{y, z, x\} + \{z, x, y\} &= 0, \\ \{\{x, y, z\}, a, b\} &= \{\{x, a, b\}, y, z\} + \{x, \{y, a, b\}, z\} + \{x, y, \{z, a, b\}\}, \end{aligned}$$

for all  $x, y, z, a, b \in L$ .

Any Lie algebra  $\mathfrak{g}$  can be considered as a Lie triple system. Indeed, one can define a ternary bracket  $\{-, -, -\} : \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  in the following way:

$$\{x, y, z\} = [x, [y, z]].$$

Then  $\mathfrak{g}$  with the bracket  $\{-, -, -\}$  is a Lie triple system.

An *ideal* in a Lie triple system  $L$  is a subspace  $L'$  such that the bracket  $\{x, y, z\}$  ( $x, y, z \in L$ ) is in  $L'$  whenever at least one of the elements  $x, y, z$  is in  $L'$ .

A Lie triple system  $L$  is called *perfect* if  $L = \{L, L, L\}$ .

Leibniz algebras were introduced recently in [5], but they already show up in many areas of mathematics. One of the aspects of their importance is the fact that they can be used to study other algebraic objects. In this article we use Leibniz algebras to study Lie triple systems and in [8] they were used to study Lie algebras. Let us recall the definition of a Leibniz algebra.

**Definition 2.** A *Leibniz algebra* is a vector space  $\mathfrak{h}$  equipped with a binary bracket

$$[-, -] : \mathfrak{h} \otimes \mathfrak{h} \rightarrow \mathfrak{h}$$

satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

for all  $x, y, z \in \mathfrak{h}$ .

Obviously, any Lie algebra is a Leibniz algebra and conversely a Leibniz algebra  $\mathfrak{g}$  is a Lie algebra provided  $[x, x] = 0$  for all  $x \in \mathfrak{g}$ . Note that, in this case, the Leibniz identity is equivalent to the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

For any Leibniz algebra, one can define a ternary bracket as it was defined above for Lie algebras. In general, we do not get a Lie triple system in this way. To get a Lie triple system in this way, it is necessary and sufficient that the Jacobi identity be satisfied in the Leibniz algebra. Such Leibniz algebras will be studied in what follows.

A beauty of Leibniz algebras is shown in the following example. Namely, one can define a Leibniz algebra structure (but not a Lie algebra structure) on second tensor and exterior powers of a Leibniz algebra or a Lie algebra. In fact, let  $\mathfrak{g}$  be a Leibniz algebra (in particular a Lie algebra). The Leibniz algebra structure on  $\mathfrak{g} \otimes \mathfrak{g}$  can be defined by

$$[x \otimes y, a \otimes b] = [x, [a, b]] \otimes y + x \otimes [y, [a, b]], \quad x, y, a, b \in \mathfrak{g}.$$

The Leibniz algebra structure on  $\mathfrak{g} \wedge \mathfrak{g}$  is defined similarly, simply replace  $\otimes$  by  $\wedge$  in the above formula to get a bracket on  $\mathfrak{g} \wedge \mathfrak{g}$ .

These constructions can be generalized to Lie triple systems. Let  $L$  be a Lie triple system. The brackets on  $L \otimes L$  and  $L \wedge L$  are given by

$$[x \otimes y, a \otimes b] = \{x, a, b\} \otimes y + x \otimes \{y, a, b\}, \quad x, y, a, b \in L,$$

$$[x \wedge y, a \wedge b] = \{x, a, b\} \wedge y + x \wedge \{y, a, b\}, \quad x, y, a, b \in L.$$

With these brackets  $L \otimes L$  and  $L \wedge L$  are Leibniz algebras.

To go further we need the definition of an action of a Leibniz algebra on a Lie triple system.

We say that a Leibniz algebra  $\mathfrak{g}$  *acts* on a Lie triple system  $L$  if we are given a map

$$L \otimes \mathfrak{g} \rightarrow L, \quad x \otimes g \mapsto x * g,$$

satisfying the following relations

$$(x * g) * h - (x * h) * g = x * [g, h],$$

$$\{x, y, z\} * g = \{x * g, y, z\} + \{x, y * g, z\} + \{x, y, z * g\},$$

for all  $x, y, z \in L$  and  $g, h \in \mathfrak{g}$ .

For any Lie triple system  $L$ , one can define an action of the Leibniz algebra  $L \wedge L$  on the Lie triple system  $L$  (in the above sense) by

$$x * (y \wedge z) = \{x, y, z\},$$

where  $x, y, z \in L$ .

Before moving to the non-abelian tensor product of Lie triple systems, let us mention an important construction which allows us to construct a Leibniz algebra structure on a vector space in some circumstances.

Let  $\mathfrak{g}$  be a Lie algebra and let  $M$  be a  $\mathfrak{g}$ -module. Assume that we are given a map of  $\mathfrak{g}$ -modules  $f : M \rightarrow \mathfrak{g}$ , where  $\mathfrak{g}$  is considered as a  $\mathfrak{g}$ -module via the adjoint representation. We call such a map a  *$\mathfrak{g}$ -equivariant map*. Then the bracket on  $M$  defined by

$$[m, n] = m * f(n), \quad m, n \in M$$

gives us a Leibniz algebra structure on  $M$ .

**1.2. A non-abelian tensor product for Lie triple systems.** Let us introduce a non-abelian tensor product for Lie triple systems. This construction allows us to describe universal central extensions of Lie triple systems explicitly.

Let  $L$  be a Lie triple system. One can equip  $L \otimes L \otimes L$  with a ternary bracket defined by

$$\{x_1 \otimes y_1 \otimes z_1, x_2 \otimes y_2 \otimes z_2, x_3 \otimes y_3 \otimes z_3\} = \{x_1, y_1, z_1\} \otimes \{x_2, y_2, z_2\} \otimes \{x_3, y_3, z_3\},$$

where  $x_i, y_i, z_i \in L$  for all  $i = 1, 2, 3$ .

Note that this bracket does not determine a Lie triple system structure. But one can take a quotient of  $L \otimes L \otimes L$  to get a Lie triple system. Let  $I$  be the subspace of  $L \otimes L \otimes L$  spanned by the elements of the forms

$$x \otimes y \otimes y,$$

$$x \otimes y \otimes z + y \otimes z \otimes x + z \otimes x \otimes y,$$

$$\{x, a, b\} \otimes y \otimes z + x \otimes \{y, a, b\} \otimes z + x \otimes y \otimes \{z, a, b\} - \{x, y, z\} \otimes a \otimes b,$$

where  $x, y, z, a, b \in L$ . One can see immediately that  $I$  is an ideal of  $L \otimes L \otimes L$  and consequently the bracket introduced above determines a bracket on the quotient  $L \otimes L \otimes L/I$ . One can straightforwardly check that the quotient with this bracket is a Lie triple system. We call this Lie triple system the non-abelian tensor cube of the Lie triple system  $L$  and denote it by  $L * L * L$ .

Note that, the original bracket of  $L$  determines a map of Lie triple systems

$$b : L * L * L \rightarrow L.$$

We define the first homology group of  $L$  (with trivial coefficients) as the cokernel of the map  $b$  and the second homology group of  $L$  (again with trivial coefficients) as the kernel of the map  $b$ . So, we have

$$H_1(L) = \text{Coker}(b),$$

$$H_2(L) = \text{Ker}(b).$$

Explicitly

$$H_1(L) = L/\{L, L, L\},$$

$$H_2(L) = \text{Ker}(\{-, -, -\})/I,$$

where  $\{-, -, -\} : L \otimes L \otimes L \rightarrow L$  is the original bracket of  $L$  and  $I$  is the subspace of  $L \otimes L \otimes L$  defined above.

**1.3. Universal central extensions of Lie triple systems.** Next, let us study the universal central extension of a Lie triple system. Let us start with the definition of central extensions of Lie triple systems.

**Definition 3.** A *central extension of a Lie triple system  $\mathfrak{g}$*  is a short exact sequence of Lie triple systems

$$0 \rightarrow K \rightarrow L \rightarrow \mathfrak{g} \rightarrow 0,$$

where  $\{L, L, K\} = 0$ .

Obviously, in a central extension we have  $\{L, K, L\} = \{K, L, L\} = 0$ .

In a usual way one can define the universal central extension of a Lie triple system and prove the following

**Proposition 1.** *Let  $L$  be a Lie triple system. The universal central extension of  $L$  exists if and only if  $L$  is perfect.*

An explicit construction of the universal central extension is obtained via the non-abelian tensor product which was introduced in the previous subsection.

Let  $L$  be a perfect Lie triple system. Then

$$b : L * L * L \rightarrow L$$

is the universal central extension of  $L$ .

This construction shows that the kernel of the universal central extension of  $L$  is the second homology group with trivial coefficients  $H_2(L)$ .

One can thus consider a perfect Lie algebra  $\mathfrak{g}$  as a Lie triple system or as a Leibniz algebra and construct the appropriate universal central extensions. One obtains three universal central extensions for each perfect Lie algebra  $\mathfrak{g}$ . Namely, let

$$(1) \quad 0 \rightarrow K_{Leib} \rightarrow U_{Leib} \rightarrow \mathfrak{g} \rightarrow 0$$

be the universal central extension of Leibniz algebra  $\mathfrak{g}$ ,

$$(2) \quad 0 \rightarrow K_{LTS} \rightarrow U_{LTS} \rightarrow \mathfrak{g} \rightarrow 0$$

be the universal central extension of Lie triple system  $\mathfrak{g}$  and

$$(3) \quad 0 \rightarrow K_{Lie} \rightarrow U_{Lie} \rightarrow \mathfrak{g} \rightarrow 0$$

be the universal central extension of Lie algebra  $\mathfrak{g}$ .

It is obvious that the universal central extension of the Lie algebra  $\mathfrak{g}$  is a quotient of the universal central extension of the Leibniz algebra (or the Lie triple system)  $\mathfrak{g}$ . There is however further unexpected relationship between these universal central extensions, which is the subject of the next section.

## 2. CENTRAL EXTENSIONS OF LIE ALGEBRAS CONSIDERED AS LIE TRIPLE SYSTEMS.

In this section we work with central extensions of perfect Lie algebras considered as Lie triple systems. Here we prove a lemma which is the key to our main result. Namely, surprisingly enough, one can construct a Leibniz algebra structure on the Lie triple system universal central extension of a perfect Lie algebra  $\mathfrak{g}$  viewed as a Lie triple system and consider it as a central extension of Leibniz algebras (see Lemma 1 below). So, the universal central extension of the Lie triple system  $\mathfrak{g}$  is a quotient of the universal central extension of the Leibniz algebra  $\mathfrak{g}$ . This leads us to the description of the universal central extension of the Lie triple system  $\mathfrak{g}$  via the universal central extension of the Leibniz algebra  $\mathfrak{g}$ . But the universal central extensions of Leibniz algebras are studied very well and thus we can use this information. The precise formulation of these facts are given in the following lemma, which as a corollary will yield our main theorem.

**Lemma 1.** *Let  $\mathfrak{g}$  be a perfect Lie algebra and let*

$$0 \rightarrow K \rightarrow L \rightarrow \mathfrak{g} \rightarrow 0$$

*be a central extension of the Lie triple system  $\mathfrak{g}$ . Then there is a Leibniz algebra structure on  $L$*

$$[-, -] : L \otimes L \rightarrow L,$$

*such that the original Lie triple system structure on  $L$  is given by the Leibniz bracket*

$$\{x, y, z\} = [x, [y, z]], \quad \forall x, y, z \in L.$$

*Proof.* We have to construct a Leibniz algebra structure on the Lie triple system  $L$ . First we construct a  $\mathfrak{g}$ -module structure on  $L$ , such that the map  $L \rightarrow \mathfrak{g}$  is a  $\mathfrak{g}$ -equivariant map. Then the desired Leibniz algebra structure can be constructed as it was described in Section 1.

It is well known that  $L \wedge L$  is a Leibniz algebra and acts on  $L$  (see Section 1). Since  $\mathfrak{g}$  is a perfect Lie algebra, it can be considered as a homomorphic image of the composition map  $L \wedge L \rightarrow \mathfrak{g} \wedge \mathfrak{g} \rightarrow \mathfrak{g}$ . Let us denote by  $Z$  the kernel of this map. We want to show that  $Z$  acts trivially on  $L$ . One can easily verify that the following statements hold:

(1) for any elements  $z \in Z$  and  $a \in L$  we have  $a * z \in K$ , where  $*$  denotes the action of  $L \wedge L$  on  $L$ ;

(2) any element  $x \in L$  can be represented as a sum of elements  $x = k + y$ , where  $k \in K$  and  $y \in \{L, L, L\}$ ;

(3) the action of  $L \wedge L$  (and in particular the action of  $Z$ ) on  $K$  is trivial.

Thus, we need to show that  $Z$  acts trivially on  $\{L, L, L\}$ . This follows from the following:

$$\begin{aligned} \{a, b, c\} * z &= \{a * z, b, c\} + \{a, b * z, c\} + \{a, b, c * z\} \\ &= \{k_1, b, c\} + \{a, k_2, c\} + \{a, b, k_3\} \\ &= 0, \end{aligned}$$

where  $k_1 = a * z, k_2 = b * z, k_3 = c * z \in K$ .

So, the action of  $L \wedge L$  on  $L$  factors through  $\mathfrak{g}$ . The desired Leibniz algebra structure on  $L$  is given by the  $\mathfrak{g}$ -equivariant map  $L \rightarrow \mathfrak{g}$ .

The identity

$$\{x, y, z\} = [x, [y, z]]$$

follows immediately from the definition of the Leibniz algebra structure on  $L$ .  $\square$

**Remark 1.** Note that the Leibniz bracket which appeared in Lemma 1 satisfies the Jacobi identity.

### 3. PROOF OF THE MAIN THEOREM

Let  $J$  be the linear subspace of  $U_{Leib}$  generated by the elements of the form

$$(4) \quad [x, [y, z]] + [z, [x, y]] + [y, [z, x]], \quad x, y, z \in U_{Leib}$$

and let  $I$  be the linear subspace of  $U_{Leib}$  generated by the elements of the form

$$[x, y] + [y, x], \quad x, y \in U_{Leib}.$$

It is obvious that  $I$  and  $J$  are subspaces of  $U_{Leib}$  and consequently ideals of  $U_{Leib}$ . Since the Jacobi identity is equivalent to the Leibniz identity modulo  $I$  and Leibniz identity holds in  $U_{Leib}$ , one can immediately conclude that  $J \subset I$ . More precise description is given by the following lemma.

**Lemma 2.** *The equation*

$$J = 2I$$

holds.

*Proof.* The elements of the form  $[x, y] + [y, x]$ ,  $x, y \in U_{Leib}$ , are in the center of  $U_{Leib}$  since their image is 0 in  $\mathfrak{g}$ . Consequently we have  $[[x, y], z] = -[[y, x], z]$  for all  $x, y, z \in U_{Leib}$ . Using this identity together with the Leibniz identity one can show the following

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= [x, [y, z]] + [[y, z], x] - [[y, x], z] + [z, [x, y]] \\ &= [x, [y, z]] + [[y, z], x] + [[x, y], z] + [z, [x, y]] \\ &= [x, [y, z]] + [[y, z], x] + [[x, z], y] + [x, [y, z]] + [z, [x, y]] \\ &= 2[x, [y, z]] + [[y, z], x] + [[x, z], y] + [z, [x, y]] \\ &= 2[x, [y, z]] + [[y, z], x] - [[z, x], y] + [z, [x, y]] \\ &= 2[x, [y, z]] + [[y, z], x] - [[z, y], x] \\ &= 2[x, [y, z]] + [[y, z], x] + [[y, z], x] \\ &= 2[x, [y, z]] + 2[[y, z], x] \\ &= 2([x, [y, z]] + [[y, z], x]). \end{aligned}$$

This proves the lemma since  $U_{Leib}$  is a perfect Leibniz algebra.  $\square$

**Corollary 1.** *We have*

$$J = 0, \quad \text{if } \text{char } F = 2,$$

$$J = I, \quad \text{if } \text{char } F \neq 2,$$

where  $F$  is the ground field.

By the universal property of (1) and considering (2) as a central extension of Leibniz algebras we get a map  $U_{Leib} \rightarrow U_{LTS}$ . Similarly we have maps  $U_{Leib} \rightarrow U_{Lie}$  and  $U_{LTS} \rightarrow U_{Lie}$ . Again by the universal property we can assemble these maps into a commutative diagram

$$\begin{array}{ccc} U_{Leib} & \longrightarrow & U_{LTS} \\ \searrow & \swarrow & \\ & U_{Lie}. & \end{array}$$

Let  $I'$  be the image of  $I$  in  $U_{LTS}$ .

**Proposition 2.** *With the above notations we have the following isomorphisms*

$$U_{LTS} \cong U_{Leib}/J$$

$$U_{Lie} \cong U_{Leib}/I, \quad \text{if } \text{char } F \neq 2,$$

$$U_{Lie} \cong U_{LTS}/I', \quad \text{if } \text{char } F \neq 2,$$

where  $F$  is the ground field. In particular

$$0 \rightarrow K_{Leib}/J \rightarrow U_{Leib}/J \rightarrow \mathfrak{g} \rightarrow 0$$

is the universal central extension of the Lie triple system  $\mathfrak{g}$ .

*Proof.* We are going to prove the first isomorphism, the other two can be proved similarly and are left to the reader as an exercise.

By Lemma 1 the extension (2) can be considered as a central extension of the Leibniz algebra  $\mathfrak{g}$ . This implies existence of the unique map  $U_{Leib} \rightarrow U_{LTS}$  over  $\mathfrak{g}$ . Moreover, the elements of the form (4) are in the kernel of this map. Recall that  $J$  is an ideal of  $U_{Leib}$ . So, the quotient  $U_{Leib}/J$  is a Leibniz algebra and the map  $U_{Leib} \rightarrow U_{LTS}$  can be factored as the composition  $U_{Leib} \rightarrow U_{Leib}/J \rightarrow U_{LTS}$ . On the other hand the ternary bracket on  $U_{Leib}/J$  defined by the formula  $\{a, b, c\} = [a, [b, c]]$  gives a Lie triple system structure and the extension

$$0 \rightarrow K_{Leib}/J \rightarrow U_{Leib}/J \rightarrow \mathfrak{g} \rightarrow 0$$

can be considered as a central extension of the Lie triple system  $\mathfrak{g}$ . So, we have the unique map  $U_{LTS} \rightarrow U_{Leib}/J$ . By the universal properties we can conclude that the maps  $U_{Leib}/J \rightarrow U_{LTS}$  and  $U_{LTS} \rightarrow U_{Leib}/J$  are inverse to each other. So, we have  $U_{LTS} \cong U_{Leib}/J$ .  $\square$

Combining the proposition above with Lemma 2 one immediately gets Theorem 1.

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